

Apolarity for determinants and permanents of generic matrices

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Abstract

We show that the apolar ideals to the determinant and permanent of a generic matrix, the Pfaffian of a generic skew symmetric matrix and the Hafnian of a generic symmetric matrix are each generated in degree two. In each case we specify the generators and a Gröbner basis of the apolar ideal. As a consequence, using a result of K. Ranestad and F. O. Schreyer we give lower bounds to the cactus rank and rank of each of these invariants. We compare these bounds with those obtained by J. Landsberg and Z. Teitler.

1 Introduction

This paper is originally motivated by a question from Zach Teitler about the generating degree of the annihilator ideal of the determinant and the permanent of a generic $n \times n$ matrix. Here annihilator is meant in the sense of the apolar pairing, i.e. Macaulay's inverse system. Our main result is that the apolar ideals of the determinant and of the permanent of a generic matrix are generated in degree 2 (Theorems 2.12 and 2.13). The reason for Teitler's interest in this problem is the recent paper by Kristian Ranestad and Frank-Olaf Schreyer [RS], which gives a lower bound for smoothable rank, border rank and cactus rank of the polynomials in terms of the generating degree of the apolar ideal and the dimension of the Artinian apolar algebra defined by the apolar ideal. We apply this and our result to bounding the scheme/cactus length of the determinant and the permanent of the generic matrix (Theorem 3.4). In section 5, we give the analogous result to those above, for the annihilator ideal of the Pfaffian of a generic skew symmetric matrix (Theorem 4.11) and the annihilator of the Hafnian of a generic symmetric matrix (Theorem 4.14).

In a sequel paper we study the apolar ideal of the determinant and permanent of the generic symmetric matrix.

Let \mathbf{k} be a field of characteristic zero or characteristic $p > 2$, and $A = (a_{ij})$ be a square matrix of size n with n^2 distinct variables. The determinant and the permanent of A are polynomials of degree n . Let $R = \mathbf{k}[a_{ij}]$ be a polynomial ring and $S = \mathbf{k}[d_{ij}]$ be the ring of inverse polynomials associated to R , and let R_k and S_k denote the degree- k homogeneous summands. Then S acts on R by contraction:

$$(d_{ij})^k \circ (a_{uv})^\ell = \begin{cases} a_{uv}^{\ell-k} & \text{if } (i, j) = (u, v), \\ 0 & \text{otherwise.} \end{cases} \quad (1)$$

This action extends multilinearly to the action of S on R . When the characteristic of the field \mathbf{k} is zero, the contraction action can be replaced by the action of partial differential operators without coefficients ([IK], Appendix A).

Definition 1.1. To each degree- j homogeneous element, $F \in R_j$ we can associate the ideal $I = \text{Ann}(F)$ in $S = \mathbf{k}[d_{ij}]$ consisting of polynomials Φ such that $\Phi \circ F = 0$. We call $I = \text{Ann}(F)$, the apolar ideal of F ; and the quotient algebra $S/\text{Ann}(F)$ the apolar algebra of F . If $h \in S_k$ and $F \in R_n$, then we have $h \circ F \in R_{n-k}$.

Let $F \in R$, then $\text{Ann}(F) \subset S$ and we have

$$(\text{Ann}(F))_k = \{h \in S_k \mid h \circ F = 0\}.$$

Remark 1.2. Let $\phi : (S_i, R_i) \rightarrow \mathbf{k}$ be the pairing $\phi(g, f) = g \circ f$, and V be a vector subspace of R_k , then we have

$$\dim_{\mathbf{k}}(V^\perp) = \dim_{\mathbf{k}} S_k - \dim_{\mathbf{k}} V. \quad (2)$$

For $V \subset R_k$, we denote by $V^\perp = \text{Ann}(V) \cap S_k$.

Let F be a form of degree j in R . We denote by $\langle F \rangle_{j-k}$ the vector space $S_k \circ F \subset R_{j-k}$. ([IK]).

We denote by $M_k(A)$ the vector subspace of R spanned by the $k \times k$ minors of A .

Lemma 1.3.

$$S_k \circ (\det(A)) = M_{n-k}(A) \subset R_{n-k}. \quad (3)$$

Proof. It is easy to see that

$$S_k \circ (\det(A)) \subset M_{n-k}(A) \subset R_{n-k}.$$

For the other inclusion, let $M_{\widehat{I},\widehat{J}}(A), I = (i_1, \dots, i_k), J = (j_1, \dots, j_k), 1 \leq i_1 \leq i_2 \leq \dots \leq i_k \leq n, 1 \leq j_1 \leq j_2 \leq \dots \leq j_k \leq n$ be the $(n-k) \times (n-k)$ minor of A one obtains by deleting the I rows and J columns of A . Now it is easy to see that $M_{\widehat{I},\widehat{J}} = \pm(d_{i_1,j_1} \cdot d_{i_2,j_2} \cdots d_{i_k,j_k}) \circ \det(A)$. Hence $M_{\widehat{I},\widehat{J}} \in S_k \circ (\det(A))$.

□

Remark 1.4. (see [IK]) Let $F \in R$ and $\deg F = j$ and $k \leq j$. Then we have

$$(\text{Ann}(F))_k = \{h \in S_k | h \circ S^{j-k}F = 0\} = (\text{Ann}(S^{j-k}F))_k. \quad (4)$$

The annihilators of the determinant and permanent of A are ideals in S , and both ideals contain all the forms of degree $n+1$ in S .

Remark 1.5. By Lemma 1.3 and Remark 1.4 we have

$$\text{Ann}(\det(A))_k = M_k(A)^\perp.$$

Example 1.6. Let $n = 3$,

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}.$$

Let P_{ij} and M_{ij} be respectively the permanent and the determinant corresponding to the entry a_{ij} . *Question:* Does $P_{11} = d_{22}d_{33} + d_{32}d_{23}$ annihilate $\det(A) = a_{11}M_{11} + a_{12}M_{12} + a_{13}M_{13}$?

$$P_{11} \circ a_{11}M_{11} = (d_{22}d_{33} + d_{23}d_{32}) \circ (a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32}) = a_{11} - a_{11} = 0.$$

$$P_{11} \circ a_{12}M_{12} = (d_{22}d_{33} + d_{23}d_{32}) \circ (a_{12}a_{21}a_{33} - a_{12}a_{23}a_{31}) = 0.$$

$$P_{11} \circ a_{13}M_{13} = (d_{22}d_{33} + d_{23}d_{32}) \circ (a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31}) = 0.$$

Hence P_{11} annihilates the determinant.

It is easy to see that when $n = 3$, $P_{ij} \circ M_{kl} = 0$ for each $1 \leq i, j, k, l \leq 3$. So in the case $n = 3$ the annihilator of the determinant of a generic matrix certainly contains all its 2×2 permanents.

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2 Two apolar algebras associated to the $n \times n$ generic matrix

2.1 Hilbert function and dimension of spaces of minors

Denote by $\mathfrak{A}_A = S/(\text{Ann}(\det(A)))$ the *apolar algebra*. Recall that the Hilbert function of \mathfrak{A}_A is defined by $H(\mathfrak{A}_A)_i = \dim_{\mathbb{k}}(\mathfrak{A}_A)_i$ for all $i = 0, 1, \dots$.

Definition 2.1. Let F be a polynomial in R , we define the $\deg(\text{Ann}(F))$ to be the length of $S/\text{Ann}(F)$.

The number of the $k \times k$ minors and permanents of a generic $n \times n$ matrix is $\binom{n}{k}^2$. The $k \times k$ minors form a linearly independent set ([BC] Theorem 5.3 and Remark 5.4), and also the $k \times k$ permanents form another linearly independent set. To see the linearly independence of these two sets we choose a term order, for example the diagonal order when the main diagonal term form a Gröbner initial term. Now the initial terms give a basis for the two spaces. So the dimension of the space of $k \times k$ minors of an $n \times n$ matrix and the dimension of the space of $k \times k$ permanents of an $n \times n$ matrix is $\binom{n}{k}^2$. By Lemma 1.3 and Remark 1.5 we have

$$H(S/\text{Ann}(\det A))_k = H(S/\text{Ann}(\text{Perm} A))_k = \binom{n}{k}^2. \quad (5)$$

So the length $\dim_{\mathbb{k}}(\mathfrak{A}_A)$ satisfies

$$\dim_{\mathbb{k}}(\mathfrak{A}_A) = \sum_{k=0}^{k=n} \binom{n}{k}^2 = \binom{2n}{n}. \quad (6)$$

2.2 Generators of the apolar ideal

In this section we determine the generators of the apolar ideal of the determinant and permanent of a generic matrix.

Notation. For a generic $n \times n$ matrix $A = (a_{ij})$, the permanent of A is a polynomial of degree n defined as follows:

$$\text{Per}(A) = \sum_{\sigma \in S_n} \Pi a_{i, \sigma(i)}$$

Lemma 2.2. *Let $A = (a_{ij})$ be a generic $n \times n$ matrix. Then each 2×2 permanent of $D = (d_{ij})$ annihilates the determinant of A .*

Proof. Assume we have an arbitrary 2×2 permanent $d_{ij}d_{kl} + d_{il}d_{kj}$.

$$P = \begin{pmatrix} d_{ij} & d_{il} \\ d_{kj} & d_{kl} \end{pmatrix}$$

$\det(A) = \sum_{\sigma \in S_n} \text{Sgn}(\sigma) \Pi a_{i, \sigma(i)}$. There are $n!$ terms in the expansion of the determinant. If a term does not contain the monomial $a_{ij}a_{kl}$ or the monomial $a_{il}a_{kj}$ then the result of action of the permanent $d_{ij}d_{kl} + d_{il}d_{kj}$ on it will be zero. There are $(n-2)!$ terms which contain the monomial $a_{ij}a_{kl}$ and $(n-2)!$ terms which contain the monomial $a_{il}a_{kj}$. So assume we have a permutation σ_1 of n objects having a_{ij} and a_{kl} respectively in its i -th and k -th place. Corresponding to σ_1 we also have a permutation $\sigma_2 = \tau\sigma_1$, where $\tau = (j, l)$ is a transposition and $\text{sgn}(\sigma_2) = \text{sgn}(\tau\sigma_1) = -\text{sgn}(\sigma_1)$. Thus corresponding to each positive term in the determinant which contains the monomial $a_{ij}a_{kl}$ or the monomial $a_{il}a_{kj}$ we have the same term with the negative sign, thus the resulting action of the permanent $d_{ij}d_{kl} + d_{il}d_{kj}$ on $\det(A)$ is zero. \square

Definition 2.3. Let $A = (a_{ij})$ and $D = (d_{ij})$ be two generic matrices with entries in the polynomial ring $R = \mathbb{k}[a_{ij}]$, and in the ring of differential operators $S = \mathbb{k}[d_{ij}]$, respectively. Let $\{\mathcal{P}_A\}$, $\{\mathcal{M}_A\}$, $\{\mathcal{P}_D\}$ and $\{\mathcal{M}_D\}$ denote the set of all 2×2 permanents and the set of all 2×2 minors of A and D , respectively. And let \mathcal{P}_A , $\mathcal{M}_A = M_2(A)$, \mathcal{P}_D and $\mathcal{M}_D = M_2(D)$ denote the spaces they span respectively.

Corollary 2.4. *Each 2×2 permanent of D annihilates \mathcal{M}_A*

Proof. By Lemma 2.1, $P_D \circ \det(A) = 0$. Let $F = \det(A)$. We have

$$(\text{Ann} F)_2 = (\text{Ann}(S_{j-2} \circ F))_2.$$

Hence

$$\mathcal{P}_D \circ \det(A) = 0 \iff \mathcal{P}_D \circ S_{j-2}(\det(A)) = 0 \iff \mathcal{P}_D \circ \mathcal{M}_A = 0.$$

\square

We also know that any square of an element, or any product of two or more elements of the same row or column of D annihilates $\det(A)$.

Definition 2.5. A monomial in the n^2 variables of the ring $S = \mathbb{k}[d_{ij}]$ is acceptable, if it is square free and has no two variables from the same row or column of D . A polynomial is acceptable if it can be written as the sum of acceptable monomials.

Lemma 2.6. $\mathcal{P}_D \oplus \mathcal{M}_D = \langle \text{degree 2 acceptable polynomials in } S \rangle$.

Proof. Let $d_{ij}d_{kl}$ be an arbitrary acceptable monomial of degree 2. Since $\text{char}(\mathbb{k}) \neq 2$ we have:

$$d_{ij}d_{kl} = 1/2((d_{ij}d_{kl} - d_{ij}d_{kl}) + (d_{ij}d_{kl} + d_{ij}d_{kl})).$$

We have (by Equation (5))

$$\dim \mathcal{P}_D = \dim \mathcal{M}_D = \binom{n}{2}^2.$$

Let $\Psi = \langle \text{degree 2 acceptable polynomials in } S \rangle$. We have

$$\dim \Psi = \dim S_2 - \dim \mathcal{U}_D = \binom{n^2 + 1}{2} - (n^2 + \binom{n}{2}(2n)).$$

So we have

$$\dim(\mathcal{P}_D + \mathcal{M}_D) = \binom{n^2 + 1}{2} - (n^2 + \binom{n}{2}(2n)) = \dim \mathcal{P}_D + \dim \mathcal{M}_D.$$

Hence $\mathcal{P}_D \cap \mathcal{M}_D = 0$.

□

Denote the space of all unacceptable polynomials of degree 2 by \mathcal{U}_D . We have shown that $(\mathcal{P}_D + \mathcal{U}_D) \circ \mathcal{M}_A = 0$ so $\mathcal{P}_D + \mathcal{U}_D \subset \text{Ann}(\mathcal{M}_A)$ and using equation (2), we know that $\dim_{\mathbb{k}}(\text{Ann}(\mathcal{M}_A))_2 = \dim_{\mathbb{k}} S_2 - \dim_{\mathbb{k}} \mathcal{M}_A$.

Lemma 2.7. $\text{Ann}(\mathcal{M}_A) \cap S_2 = \mathcal{P}_D + \mathcal{U}_D$.

Proof. By Lemma 2.5 we have $\mathcal{P}_D + \mathcal{M}_D$ is complementary to \mathcal{U}_D . So we have

$$\dim((\text{Ann}(\mathcal{M}_A))_2) = \dim S_2 - \dim \mathcal{M}_A = \dim \mathcal{P}_D + \dim \mathcal{U}_D.$$

□

We define the homomorphism $\xi : R \rightarrow S$ by setting $\xi(a_{ij}) = d_{ij}$; for a monomial $v \in R$ we denote by $\hat{v} = \xi(v)$ the corresponding monomial of S .

Remark 2.8. Let $f = \sum_{i=1}^{i=k} \alpha_i v_i \in R_n$ with $\alpha_i \in \mathbf{k}$ and with v_i 's linearly independent monomials. Then we will have:

$$\text{Ann}(f) \cap S_n = \langle \alpha_j \hat{v}_1 - \alpha_1 \hat{v}_j, \langle v_1, \dots, v_k \rangle^\perp \rangle, \quad (7)$$

where $\langle v_1, \dots, v_k \rangle^\perp = \text{Ann}(\langle v_1, \dots, v_k \rangle) \cap S_n$.

Lemma 2.9.

$$(\mathcal{P}_D + \mathcal{U}_D)_k \subset \text{Ann}(M_k(A)) \cap S_k. \quad (8)$$

Proof. We have:

$$(1) \mathcal{P}_D \circ \det(A) = 0 \iff \mathcal{P}_D \circ S_{n-2}(\det(A)) = 0 \iff \mathcal{P}_D \circ \mathcal{M}_A = 0.$$

$$(2) (\text{Ann}(\det(A))) \cap S_2 = \mathcal{P}_D + \mathcal{U}_D \Rightarrow S_{k-2}(\mathcal{P}_D + \mathcal{U}_D) \circ (S_{n-k} \circ \det(A)) = 0.$$

$$\Rightarrow S_{k-2}(\mathcal{P}_D + \mathcal{U}_D) \circ M_k(A) = 0.$$

$$\Rightarrow (\mathcal{P}_D + \mathcal{U}_D)_k \circ M_k(A) = 0. \text{ (By Remark 1.4)}$$

Therefore Equation (8) holds. □

Proposition 2.10. For a generic $n \times n$ matrix A where $n \geq 2$, we have

$$(\mathcal{P}_D + \mathcal{U}_D)_n = \text{Ann}(\det(A)) \cap S_n.$$

Proof. Using Equation (8) we only need to show

$$(\mathcal{P}_D + \mathcal{U}_D)_n \supset \text{Ann}(\det(A)) \cap S_n.$$

We use induction on n . For $n = 2$ the equality is easy to see. First we show the proposition holds for the case $n = 3$. We need to show, the space of 2×2 permanents of D generates $\text{Ann}(\det(A))_3/\mathcal{U}_D$ i.e. $\text{Ann}(M_3(A))_3/\mathcal{U}_D$. Corresponding to each term in the determinant, there is a permutation of three objects σ such that we can write the term as $a_{1\sigma(1)}a_{2\sigma(2)}a_{3\sigma(3)}$. Now consider the degree three binomial $b = a_{1\sigma(1)}a_{2\sigma(2)}a_{3\sigma(3)} - a_{1\tau(1)}a_{2\tau(2)}a_{3\tau(3)}$, where $\tau \neq \sigma$. Without loss of generality we can assume that σ is the identity, so we consider the binomial $b = a_{11}a_{22}a_{33} - a_{1\tau(1)}a_{2\tau(2)}a_{3\tau(3)}$. If these two monomials have a common variable, i.e $\tau(i) = i$ for some $i = 1, 2, 3$ then the binomial will be of the form $b = a_{ii}(a_{jj}a_{kk} - a_{jk}a_{kj})$, $1 \leq i, j, k, l \leq 3$. so we will have $b = a_{ii}M_{ii}$ and as we have shown previously $P_{ii} = d_{jj}d_{kk} - d_{jk}d_{kj}$ annihilates it. Now assume that the

monomials $a_{11}a_{22}a_{33}$ and $a_{1\tau(1)}a_{2\tau(2)}a_{3\tau(3)}$ do not have any common factor, we can add and subtract another term $a_{1\beta(1)}a_{2\beta(2)}a_{3\beta(3)}$, where β is a permutation, such that it has one common factor with $a_{11}a_{22}a_{33}$ and one common factor with $a_{1\tau(1)}a_{2\tau(2)}a_{3\tau(3)}$. Without loss of generality we can take $\beta(1) = \tau(1)$, $\beta(2) = 2$ and then we can determine $\beta(3)$ according to the other two choices. Then by factorizing we get a binomial of the form $a_{ij}M_{ij} + a_{kl}M_{kl}$, where the first term can be annihilated by the permanent of the matrix D corresponding to d_{ij} and the second term can be annihilated by the permanent of the matrix D corresponding to the element d_{kl} . So by Equation (7) we are done. As an example, If we have the binomial $a_{11}a_{22}a_{33} - a_{13}a_{21}a_{32}$ we can add and subtract the term $a_{11}a_{23}a_{32}$ which has one common factor with $a_{11}a_{22}a_{33}$ and one common factor with $a_{13}a_{21}a_{32}$ so we will get $a_{11}(a_{22}a_{33} - a_{23}a_{32}) + a_{32}(a_{11}a_{23} - a_{13}a_{21})$ which is $a_{11}M_{11} + a_{32}M_{32}$. And as we have shown before it can be annihilated by the space of 2×2 permanents. So by Equation (7) we are done.

When n is larger than 3 by the induction assumption we can assume that the proposition holds for all $k \leq n - 1$. By the Remark 2.7 it is enough to show that if b is a binomial of the form (7), in $\text{Ann}(\det(A)) \cap S_n$, then $b \in (\mathcal{P}_D + \mathcal{U}_D)_n$. Assume $b = b_1 + b_2$ is of degree n . If the two terms, b_1 and b_2 are monomials in S and have a common factor l , i.e. $b_1 = la_1$ and $b_2 = la_2$, then $b = l(a_1 + a_2)$ where a_1 and a_2 are of degree at most $n - 1$. Now by the induction assumption the proposition holds for the binomial $a_1 + a_2$, i.e. $a_1 + a_2 \in (\mathcal{P}_D + \mathcal{U}_D)_{n-1}$ hence we have

$$b = l(a_1 + a_2) \in l(\mathcal{P}_D + \mathcal{U}_D)_{n-1} \subset (\mathcal{P}_D + \mathcal{U}_D)_n.$$

If the two terms, b_1 and b_2 do not have any common factor then with the same method as above we can rewrite the binomial b by adding and subtracting a term of the determinant, m of degree n , which has a common factor m_1 with b_1 and a common factor m_2 with b_2 , now we will have

$$b_1 + b_2 = b_1 + m + b_2 - m = m_1(c_1 + m') + m_2(c_2 - m''),$$

where $b_1 = m_1c_1$, $m = m_1m' = m_2m''$ and $b_2 = m_2c_2$. Now $c_1 + m'$ and $c_2 - m''$ are of degree at most $n - 1$ so by the induction assumption we have

$$b_1 + b_2 = m_1(c_1 + m') + m_2(c_2 - m'') \in (\mathcal{P}_D + \mathcal{U}_D)_n.$$

This completes the induction step and the proof of the proposition. □

Corollary 2.11. *For a generic $n \times n$ matrix A and each k , $1 \leq k \leq n$, we have*

$$(\mathcal{P}_D + \mathcal{U}_D)_k = \text{Ann}(\det(A)) \cap S_k.$$

We also have $(\mathcal{U}_D)_{n+1} = S_{n+1}$.

Proof. Using equation (8) we only need to show that

$$\text{Ann}(\det(A)) \cap S_k \subset (\mathcal{P}_D + \mathcal{U}_D)_k.$$

By Lemma 1.3 and Remark 1.4 we have

$$(\text{Ann}(\det(A)))_k = (\text{Ann}(S_{n-k} \circ (\det(A))))_k = (\text{Ann}(M_k(A)))_k$$

Now if we label the $k \times k$ minors of A by f_1, \dots, f_s we have

$$(\text{Ann}(M_k(A)))_k = \text{Ann}(\langle f_1, \dots, f_s \rangle)_k = \left(\bigcap_{i=1}^{i=s} \text{Ann}(f_i) \right)_k$$

But for each f_i if we denote the ring of variables of f_i by R^i by Proposition 2.10 we have

$$(\mathcal{P}_D^i + \mathcal{U}_D^i)_k = \text{Ann}(f_i) \cap S_k^i$$

So we have

$$\text{Ann}(\det(A)) \cap S_k \subset (\mathcal{P}_D + \mathcal{U}_D)_k.$$

Every monomial of degree larger than n will be unacceptable. So we have $(\mathcal{U}_D)_{n+1} = S_{n+1}$. □

Theorem 2.12. *Let A be a generic $n \times n$ matrix. Then the apolar ideal $\text{Ann}(\det(A)) \subset S$ is the ideal $(\mathcal{P}_D + \mathcal{U}_D)$, generated in degree two.*

Proof. This follows directly from the Lemmas 2.1, 2.6 and the Proposition 2.10. □

Theorem 2.13. *Let A be a generic $n \times n$ matrix. Then the apolar ideal $\text{Ann}(\text{Per}(A)) \subset S$ to $\text{Per}(A) \in R$ is the ideal $(\mathcal{M}_D + \mathcal{U}_D)$, generated in degree two.*

Proof. The proof follows directly from the proof of the Lemmas 2.1, 2.6 and the Proposition 2.10, by interchanging the determinants and the permanents. □

Corollary 2.14. *Let $A = (a_{ij})$ be an $m \times n$ matrix where $n \geq m$. Let N denote the space generated by all $m \times m$ minors of A . Then $\text{Ann}(N)$ is generated in degree two, by all 2×2 permanents of A and the degree two unacceptable monomials.*

Proof. Let $s = \binom{n}{m}$, and f_1, \dots, f_s denote the $m \times m$ minors of A . We have

$$\text{Ann}(N) = \text{Ann}(\langle f_1, \dots, f_s \rangle) = \bigcap_{i=1}^{i=s} (\text{Ann}(f_i)).$$

Let R^i denote the ring of variables of f_i . Hence by Theorem 2.12 we have $\text{Ann}(f_i) \cap S^i$ is generated in degree 2. So we have $\text{Ann}(N)$ is also generated in degree 2. □

3 Application to the ranks of the determinant and the permanent

Let $F \in R = \mathbb{k}[a_{ij}]$ be a homogeneous form. The apolarity action of $S = \mathbb{k}[d_{ij}]$ on R , defines S as a natural coordinate ring on the projective space $\mathbf{P}(T_1)$ of 1-dimensional subspaces of T_1 . A finite subscheme $\Gamma \subset \mathbf{P}(T_1)$ is apolar to F if the homogeneous ideal $I_\Gamma \subset S$ is contained in $\text{Ann}(F)$ ([IK],[RS]).

Definition 3.1. We have the following ranks ([IK] Def. 5.66 , [BR] and [RS])

a. the cactus rank (scheme rank) $cr(F)$:

$$cr(F) = \min\{\deg \Gamma \mid \Gamma \subset \mathbf{P}(T_1), \dim \Gamma = 0, I_\Gamma \subset \text{Ann}(F)\}.$$

b. the smoothable rank $sr(F)$:

$$sr(F) = \min\{\deg \Gamma \mid \Gamma \subset \mathbf{P}(T_1) \text{ smoothable}, \dim \Gamma = 0, I_\Gamma \subset \text{Ann}(F)\}.$$

c. the rank $r(F)$:

$$r(F) = \min\{\deg \Gamma \mid \Gamma \subset \mathbf{P}(T_1) \text{ smooth}, \dim \Gamma = 0, I_\Gamma \subset \text{Ann}(F)\}.$$

d. the differential rank (Sylvester's catalecticant or apolarity bound):

$$l_{diff}(F) = \max\{H(S/\text{Ann}(F))_i\}.$$

Proposition 3.2. ([IK], Proposition 6.7C) *The above ranks satisfy*

$$l_{diff}(F) \leq cr(F) \leq sr(F) \leq r(F).$$

Proposition 3.3. (*Ranestad-Schreyer*) *If the ideal of $\text{Ann}(F)$ is generated in degree d and $\Gamma \subset \mathbf{P}(T_1)$ is a finite (punctual) apolar subscheme to F , then*

$$\deg \Gamma \geq \frac{1}{d} \deg(\text{Ann}(F)),$$

where $\deg(\text{Ann}(F)) = \dim(S/\text{Ann}(F))$ is the length of the 0-dimensional scheme defined by $\text{Ann}(F)$.

Now if we take $F = \det(A)$ or $F = \text{Per}(A)$, since we have found that for the determinant and the permanent of a matrix, we have $d = 2$; we can use the above proposition to find a lower bound for the above ranks of F .

Theorem 3.4. *Let F be the determinant or permanent of a generic $n \times n$ matrix A . We have*

$$\frac{1}{2} \binom{2n}{n} \leq cr(F) \leq sr(F) \leq r(F).$$

Proof. By Theorems 2.12, 2.13, Proposition 3.3, Equations 5 and equation 6 we have for an apolar punctual scheme Γ ,

$$\deg \Gamma \geq \frac{1}{d} \deg(\text{Ann}(F)) = \frac{1}{2} \sum_{k=0}^{k=n} \binom{n}{k}^2 = \frac{1}{2} \binom{2n}{n}.$$

□

Notation. [LT] Let $\Phi \in S^d \mathbb{C}^n$ be a polynomial, we can polarize Φ and consider it as a multilinear form $\tilde{\Phi}$ where $\Phi(x) = \tilde{\Phi}(x, \dots, x)$ and consider the linear map $\Phi_{s,d-s} : S^s \mathbb{C}^{n*} \rightarrow S^{d-s} \mathbb{C}^n$, where $\Phi_{s,d-s}(x_1, \dots, x_s)(y_1, \dots, y_{d-s}) = \tilde{\Phi}(x_1, \dots, x_s, y_1, \dots, y_{d-s})$. Define

$$\text{Zeros}(\Phi) = \{[x] \in \mathbb{P}\mathbb{C}^{n*} | \Phi(x) = 0\} \subset \mathbb{P}\mathbb{C}^{n*}.$$

Let x_1, \dots, x_n be linear coordinates on \mathbb{C}^{n*} and define

$$\Sigma_s(\Phi) := \{[x] \in \text{Zeros}(\Phi) | \frac{\partial^I \Phi}{\partial x^I}(x) = 0, \forall I, \text{ such that } |I| \leq s\}.$$

In our notation $\Phi_{s,d-s}$ is the map from $S_s \rightarrow R_{n-s}$ taking h to $h \circ \Phi$, hence its rank is $H(\mathfrak{A}_A)_s$.

In the following theorem we use the convention that $\dim \emptyset = -1$.

Theorem 3.5. (*Landsberg-Teitler*) (*[LT]*) Let $\Phi \in S^d \mathbb{C}^n$, Let $1 \leq s \leq d$. Then

$$\text{rank}(\Phi) \geq \text{rank} \Phi_{s,d-s} + \dim \Sigma_s(\Phi) + 1.$$

Remark. (Z. Teitler) If we define $\Sigma_s(\Phi)$ to be a subset of affine, rather than projective space then the above theorem does not need +1 at the end, and does not need the statement that the dimension of the empty set is -1 .

Applying this theorem for the determinant yields

Corollary 3.6. (*Landsberg-Teitler*) (*[LT]*)

$$r(\det_n) \geq \binom{n}{\lfloor n/2 \rfloor}^2 + n^2 - (\lfloor n/2 \rfloor + 1)^2.$$

Proposition 3.7. (*Bernardi-Ranestad*) (*[BR]*) Let $F \in R^s$ be a homogeneous form of degree d , and let l be any linear form in S_1^s . Let F_l be a dehomogenization of F with respect to l . Denote by $\text{Diff}(F)$ the subspace of S^s generated by the partials of F of all orders. Then

$$cr(F) \leq \dim_{\mathbf{k}} \text{Diff}(F_l)$$

Proposition 3.8. (*[CCG]*) (a) For the monomial $x_1^{b_1} \dots x_n^{b_n}$, where $1 \leq b_1 \leq \dots \leq b_n$ we have

$$r(x_1^{b_1} \dots x_n^{b_n}) = \prod_{i=2}^n (b_i + 1)$$

(b) (*[RS]*) For the monomial $x_1^{b_1} \dots x_n^{b_n}$, where $1 \leq b_1 \leq \dots \leq b_n$ we have

$$sr(x_1^{b_1} \dots x_n^{b_n}) = cr(x_1^{b_1} \dots x_n^{b_n}) = \prod_{i=1}^{n-1} (b_i + 1)$$

Example 3.9. Let $n = 2$, and

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

$$\det(A) = ad - bc = (a + d)^2 - (a - d)^2 + (b - c)^2 - (b + c)^2 \text{ so } r(\det(A)) = 4.$$

The corresponding Hilbert sequence for $n = 2$ is $(1, 4, 1)$. We have $l_{diff}(\det(A)) = 4$. Now using Theorem 3.4 we have:

$$cr(\det(A)) \geq \frac{1}{d} \deg(\text{Ann}(\det(A))) = \frac{1}{2}(6) = 3.$$

So the lower bound we can find using Theorem 3.4 is 3.

Now using Corollary 3.6 (Landsberg-Teitler) we will have:

$$r(\det_2) \geq \binom{2}{\lfloor 2/2 \rfloor}^2 + 2^2 - (\lfloor 2/2 \rfloor + 1)^2 = 4 + 4 - 4 = 4.$$

On the other hand we have

$$\det(A) = ad - bc = 1/4((a + d)^2 - (a - d)^2) - 1/4((b + c)^2 - (b - c)^2).$$

So we have

$$r(\det(A)) = cr(\det(A)) = sr(\det(A)) = l_{diff}(\det(A)) = 4.$$

Example 3.10. Let $n = 3$, and

$$A = \begin{pmatrix} a & b & e \\ c & d & f \\ g & h & i \end{pmatrix},$$

$$\det(A) = g(bf - de) - h(af - ce) + i(ad - bc).$$

The corresponding Hilbert sequence in the case $n = 3$ is $(1, 9, 9, 1)$. Now using Theorem 3.4 we have:

$$cr(\det(A)) \geq \frac{1}{d} \deg(\text{Ann}(\det(A))) = \frac{1}{2}(20) = 10.$$

So the lower bound we can find using the Theorem 3.4 is 10, which is greater than the $l_{diff}(\det(A)) = 9$. So it is a better lower bound for the cactus and smoothable ranks introduced above.

Using Corollary 3.6 we have:

$$r(\det_3) \geq \binom{3}{\lfloor 3/2 \rfloor}^2 + 3^2 - (\lfloor 3/2 \rfloor + 1)^2 = 9 + 9 - 4 = 14.$$

So the lower bound of the rank given by Corollary 3.6 (Landsberg-Teitler) is a better lower bound for the rank in this case.

On the other hand, for every x, y and z , it is easy to see that $r(xyz) \leq 4$:

$$xyz = 1/24((x+y+z)^3 + (x-y-z)^3 - (x-y+z)^3 - (x+y-z)^3).$$

Hence $14 \leq r(\det(A)) \leq 24$.

Now let $a = 1$ in $\det(A)$, we have that the punctual scheme $\text{Ann}(\det A_{a=1})$ of Hilbert function $(1, 9, 17, 18, 18, \dots)$ and degree 18 is apolar to $\det A$. So by Proposition 3.7 we have:

$$cr(\det(A)) \leq 18.$$

Example 3.11. Let $n = 4$, and

$$A = \begin{pmatrix} a & b & e & j \\ c & d & f & k \\ g & h & i & l \\ m & n & o & p \end{pmatrix},$$

The corresponding Hilbert sequence in the case $n = 4$ is $(1, 16, 36, 16, 1)$. By Theorem 3.4,

$$cr(\det(A)) \geq \frac{1}{d} \deg(\text{Ann}(\det(A))) = \frac{1}{2}(70) = 35.$$

which is less than the $l_{diff}(\det(A)) = 36$. So in this case l_{diff} is a better lower bound for the cactus rank.

Using Corollary 3.6 (Landsberg-Teitler) we have:

$$r(\det_4) \geq \binom{4}{\lfloor 4/2 \rfloor}^2 + 4^2 - (\lfloor 4/2 \rfloor + 1)^2 = 36 + 16 - 9 = 43.$$

So the lower bound found by Corollary 3.6 (Landsberg-Teitler) is a better lower bound for the rank in this case.

Now using Proposition 3.8 we have

$$r(\det_4) \leq (4!)(2^3) = 192$$

Example 3.12. Let $n = 5$, and

$$A = \begin{pmatrix} a & b & e & j & q \\ c & d & f & k & r \\ g & h & i & l & s \\ m & n & o & p & t \\ u & v & w & x & y \end{pmatrix},$$

The corresponding Hilbert sequence in the case $n = 5$ is $(1, 25, 100, 100, 25, 1)$. By Theorem 3.4

$$cr(\det(A)) \geq \frac{1}{d} \deg(\text{Ann}(\det(A))) = \frac{1}{2}(252) = 126,$$

which is greater than the $l_{diff}(\det(A)) = 100$. So it is a better lower bound for cactus rank than l_{diff} .

Using Corollary 3.6 (Landsberg-Teitler) we have:

$$r(\det_5) \geq \binom{5}{\lfloor 5/2 \rfloor}^2 + 5^2 - (\lfloor 5/2 \rfloor + 1)^2 = 116.$$

So for the first time at $n = 5$ Theorem 3.4 gives us a better lower bound for the rank than Corollary 3.6 (Landberg-Teitler).

Now using Proposition 3.8 we have

$$r(\det_5) \leq (5!)(2^4) = 1920$$

Example 3.13. The corresponding Hilbert sequence when $n = 6$ is

$$H(S/\text{Ann}(\det A)) = (1, 36, 225, 400, 225, 36, 1).$$

Now using Corollary 3.4 we have:

$$cr(\det(A)) \geq \frac{1}{d} \deg(\text{Ann}(\det(A))) = \frac{1}{2}(924) = 462.$$

So the lower bound we can find using the Theorem 3.4 is 462, which is greater than the $l_{diff}(\det(A)) = 400$. So it is a better lower bound for cactus rank than l_{diff} .

Using Corollary 3.6 (Landsberg-Teitler) we have:

$$r(\det_6) \geq \binom{6}{\lfloor 6/2 \rfloor}^2 + 6^2 - (\lfloor 6/2 \rfloor + 1)^2 = 420.$$

So again at $n = 6$ Theorem 3.4 give us a better lower bound than Corollary 3.6 (Landberg-Teitler).

Now using Proposition 3.8 we have

$$r(\det_6) \leq (6!)(2^5) = 23040$$

Table 1: The determinant of the generic matrix

n	2	3	4	5	6	$n \gg 0$
lower bound for $cr(\det(A))$ by Theorem 3.4	3	10	35	126	462	$4^n/2\sqrt{n\pi}$
lower bound for $r(\det(A))$ by Corollary 3.6	4	14	43	116	420	$4^n/2n\pi$
$l_{diff}(\det(A))$	4	9	36	100	400	$\left(\binom{n}{\lfloor n/2 \rfloor}\right)^2$

Remark 3.14. (a) Using Stirling's formula, $n! \sim \sqrt{2\pi n} \left(\frac{n}{e}\right)^n$, we can approximate $\binom{2n}{n}$ for large n by $4^n/\sqrt{n\pi}$. Hence for large n Theorem 3.4 gives us the lower bound asymptotic to $4^n/2\sqrt{n\pi} \leq cr(\det(A))$, and Landsberg-Teitler formula give us the lower bound $2.4^n/(n\pi) \leq r(\det(A))$. This bound is asymptotic also to $l_{diff}(\det(A))$, a lower bound for $cr(\det(A))$.

(b) Using Proposition 3.8 the upper bound for the rank of a generic $n \times n$ matrix is given by $(n!)2^{n-1}$. This can be approximated for large n , using Stirling's formula, by $\sqrt{2\pi n} \left(\frac{n}{e}\right)^n (2^{n-1})$.

In the following table we give the lower bounds for the rank of the determinant of an $n \times n$ generic matrix.

4 Annihilator of the Pfaffian and Hafnian

In this section we discuss the annihilator ideals of the Pfaffians and of the Hafnians. We show that the annihilator ideal of the Pfaffian of a generic skew symmetric $2n \times 2n$ matrix and the annihilator ideal of the Hafnian of generic symmetric $2n \times 2n$ matrix are generated in degree 2.

In the following discussion we let $X_m^{sk} = (x_{ij})$ with $x_{ij} = -x_{ji}$ be an $m \times m$ skew symmetric matrix of indeterminates in the polynomial ring $R^{sk} = k[x_{ij}]$, Let $Y_m^{sk} = (y_{ij})$ with $y_{ij} = -y_{ji}$ be an $m \times m$ skew symmetric matrix of indeterminates in the ring of differential operators $S^{sk} = k[y_{ij}]$. We denote the Pfaffian of the matrix X_m^{sk} by $Pf(X_m^{sk})$. It is well known that for any odd number m we have $\det(X_m^{sk}) = 0$. It is also well known that the square of the Pfaffian directly yields the value of the determinant of a skew symmetric matrix. So in the following we are going to consider the annihilator of the Pfaffian of the generic $m \times m$ skew symmetric matrices, where $m = 2n$ is an even number.

Notation. Let $F_{2n} \subset S_{2n}$ is the set of all permutations σ satisfying the following conditions:

- (1) $\sigma(1) < \sigma(3) < \dots < \sigma(2n-1)$

(2) $\sigma(2i-1) < \sigma(2i)$ for all $1 \leq i \leq n$

- For a $2n \times 2n$ generic skew symmetric matrix X^{sk} , we denote by $Pf(X^{sk})$ the Pfaffian of X^{sk} defined as

$$Pf(X^s) = \sum_{\sigma \in F_{2n}} \text{sgn}(\sigma) x_{\sigma(1)\sigma(2)} x_{\sigma(3)\sigma(4)} \cdots x_{\sigma(2n-1)\sigma(2n)} \quad (9)$$

- ([IKO]) We denote by $Hf(X^s)$ the Hafnian of a generic symmetric $2n \times 2n$ matrix X^s , defined by

$$Hf(X^s) = \sum_{\sigma \in F_{2n}} x_{\sigma(1)\sigma(2)} x_{\sigma(3)\sigma(4)} \cdots x_{\sigma(2n-1)\sigma(2n)} \quad (10)$$

Let $J_{2n} = \text{Ann}(Pf(X_{2n}^{sk}))$. We first give some examples and then some partial results concerning $\text{Ann}(Pf(X_{2n}^{sk}))$. Using Macaulay2 for calculations we have the following results:

(a) Let X_2 be a generic skew symmetric 2×2 matrix, then we have $H(S^{sk}/J_2) = (1, 1)$. And the maximum degree of the generators of the annihilator ideal J_2 is 2. Now using the Ranestad-Schreyer Proposition we have:

$$cr(Pf(X_2^{sk})) \geq \frac{1}{d} \deg(\text{Ann}(Pf(X_2^{sk}))) = \frac{1}{2}(2) = 1,$$

which is the same as the differential length in this case. In this case $r(Pf(X_2^{sk}))=1$ evidently, so we have

$$r(Pf(X_2^{sk})) = cr(Pf(X_2^{sk})) = sr(Pf(X_2^{sk})) = l_{diff}(Pf(X_2^{sk})) = 1.$$

(b) Let X_4 be a generic skew symmetric 4×4 matrix, we have $H(S^{sk}/J_4) = (1, 6, 1)$. And the maximum degree of the generators of the annihilator ideal J_4 is 2. Using the Ranestad-Schreyer Proposition we have:

$$cr(Pf(X_4^{sk})) \geq \frac{1}{d} \deg(\text{Ann}(Pf(X_4^{sk}))) = \frac{1}{2}(8) = 4,$$

which is less than $l_{diff} = 6$.

(c) Let X_6 be a generic skew symmetric 6×6 matrix, we have $H(S^{sk}/J_6) = (1, 15, 15, 1)$. And the maximum degree of the generators of the annihilator ideal J_6 is 2. Using the Ranestad-Schreyer Proposition we have:

$$cr(Pf(X_6^{sk})) \geq \frac{1}{d} \deg(\text{Ann}(Pf(X_6^{sk}))) = \frac{1}{2}(32) = 16,$$

which is larger than $l_{diff} = 15$.

(d) Let X_8 be a generic skew symmetric 8×8 matrix, we have $H(S^{sk}/J_8) = (1, 28, 70, 28, 1)$. And the maximum degree of the generators of the annihilator ideal J_8 is 2. Now using the Ranestad-Schreyer Proposition we have:

$$cr(Pf(X_8^{sk})) \geq \frac{1}{d} \deg(\text{Ann}(Pf(X_8^{sk}))) = \frac{1}{2}(128) = 64,$$

which is less than $l_{diff} = 70$.

(e) Let X_{10} be a generic skew symmetric 10×10 matrix, we have

$$H(S^{sk}/J_{10}) = (1, 45, 210, 210, 45, 1).$$

And the maximum degree of the generators of the annihilator ideal J_{10} is 2. Now using the Ranestad-Schreyer Proposition we have:

$$cr(Pf(X_{10}^{sk})) \geq \frac{1}{d} \deg(\text{Ann}(Pf(X_{10}^{sk}))) = \frac{1}{2}(512) = 256,$$

which is larger than $l_{diff} = 210$.

Remark 4.1. The Hilbert sequence for the apolar algebra of Pfaffian of each $2n \times 2n$ matrix is given by $\binom{2n}{2t}$. and we have $\sum_{t=0}^{t=n} \binom{2n}{2t} = 2^{2n-1}$.

Definition 4.2. A $2t$ -Pfaffian minor of a skew symmetric matrix X is a Pfaffian of a submatrix of X consisting of rows and columns indexed by i_1, i_2, \dots, i_{2t} for some $i_1 < i_2 < \dots < i_{2t}$.

The number of $2t$ -Pfaffian minors of a $2n \times 2n$ skew symmetric matrix is clearly $\binom{2n}{2t}$. We denote by $\{P_{2t}(X^{sk})\}$ the set of the $2t$ -Pfaffians of X^{sk} . Furthermore, we denote by $P_{2t}(X^{sk})$ the vector space generated by $\{P_{2t}(X^{sk})\}$ in R_t^{sk} and we denote by $(P_{2t}(X^{sk}))$ the ideal generated by $\{P_{2t}(X^{sk})\}$ in R^{sk} . Let τ be the lexicographic term order on $R^{sk} = \mathbb{k}[x_{ij}]$ induced by the following order on the indeterminates:

$$x_{1,2n} \geq x_{1,2n-1} \geq \dots \geq x_{1,2} \geq x_{2,2n} \geq x_{2,2n-1} \geq \dots \geq x_{2n-1,2n}.$$

Theorem 4.3. (Herzog-Trung [HT], Theorem 4.1) The set $\{P_{2t}(X)\}$ of the $2t$ -Pfaffians of the matrix X^{sk} is a Gröbner basis of the ideal $(P_{2t}(X))$ with respect to τ .

Corollary 4.4. *The dimension of the space of $2t \times 2t$ Pfaffians of a $2n \times 2n$ generic skew symmetric matrix X^{sk} is $\binom{2n}{2t}$. So we have*

$$\dim(S^{sk}/\text{Ann}(Pf(X^{sk}))) = 2^{2n-1}.$$

Proof. The proof follows directly from the Theorem 4.3 and the fact that

$$\sum_{t=0}^{t=n} \binom{2n}{2t} = 2^{2n-1}.$$

□

In the remaining part of this section we prove that the annihilator ideal of the Pfaffian of a generic skew symmetric matrix is generated in degree 2.

Definition 4.5. Let W be the vector subspace of S^{sk} spanned by degree 2 elements of type (a), (b) and (c) defined as follows

- (a) square of each element of Y^{sk} . The number of these monomials is $2n^2 - n$.
- (b) product of each element of Y^{sk} with another element in the same row or column of the matrix Y^{sk} . The number these monomials is $(2n^2 - n)(2n - 2)$.
- (c) Given any 4×4 submatrix of X^{sk} of the rows and columns i_1, i_2, i_3 and i_4 ,

$$Q = \begin{pmatrix} 0 & x_{i_1 i_2} & x_{i_1 i_3} & x_{i_1 i_4} \\ -x_{i_1 i_2} & 0 & x_{i_2 i_3} & x_{i_2 i_4} \\ -x_{i_1 i_3} & -x_{i_2 i_3} & 0 & x_{i_3 i_4} \\ -x_{i_1 i_4} & -x_{i_2 i_4} & -x_{i_3 i_4} & 0 \end{pmatrix},$$

we have $Pf(Q) = x_{i_1 i_2} x_{i_3 i_4} - x_{i_1 i_3} x_{i_2 i_4} + x_{i_1 i_4} x_{i_2 i_3}$. Corresponding to $Pf(Q)$ we have 3 binomials which annihilate $Pf(Q)$ hence annihilate $Pf(X^{sk})$. These binomials are $y_{i_1 i_2} y_{i_3 i_4} + y_{i_1 i_3} y_{i_2 i_4}$, $y_{i_1 i_2} y_{i_3 i_4} - y_{i_1 i_4} y_{i_2 i_3}$ and $y_{i_1 i_3} y_{i_2 i_4} + y_{i_1 i_4} y_{i_2 i_3}$. But evidently these three binomials are not linearly independent, and we can write one of them as the sum of the other 2 binomials. So corresponding to each 4×4 Pfaffian we have 2 linearly independent binomials in the annihilator ideal. So using Theorem 4.3, the number of these binomials will be $2\binom{2n}{4}$.

Remark. For a $2n \times 2n$ skew symmetric matrix X^{sk} , $W \subset \text{Ann}(Pf(X^{sk}))$.

Lemma 4.6. *For the generic skew symmetric $2n \times 2n$ matrix X^{sk} , we have*

$$W = \text{Ann}(P_4(X^{sk})) \cap S_2^{sk}.$$

Proof. The monomials of type (a) and (b) correspond to unacceptable monomials discussed earlier and are linearly independent from any binomial in (c). And the binomials in (c) are linearly independent by Theorem 4.3. Hence we have

$$\dim_k(W) = 2 \binom{2n}{4} + (2n^2 - n)(2n - 2) + 2n^2 - n = \binom{2n^2 - n + 1}{2} - \binom{2n}{4}. \quad (11)$$

According to Remark 1.2 we have

$$\dim_k(\text{Ann}(P_4(X^{sk}))) \cap S_2^{sk} = \dim_k S_2^{sk} - \dim_k(P_4(X^{sk})).$$

So we have

$$\dim_k(\text{Ann}(P_4(X^{sk}))) \cap S_2^{sk} = \binom{2n^2 - n + 1}{2} - \binom{2n}{4}. \quad (12)$$

Using equations (11) and (12) we have

$$\dim_k(W) = \dim_k(\text{Ann}(P_4(X^{sk}))) \cap S_2^{sk}. \quad (13)$$

On the other hand, evidently we have

$$W \subset \text{Ann}(P_4(X^{sk})) \cap S_2^{sk}. \quad (14)$$

Using equations (13) and (14) we have

$$W = \text{Ann}(P_4(X^{sk})) \cap S_2^{sk}.$$

□

Lemma 4.7. *Let X^{sk} be a $2n \times 2n$ skew symmetric matrix ($n \geq 2$). We have,*

$$S_{n-2} \circ Pf(X^{sk}) = P_4(X^{sk}) \subset R_2^{sk}.$$

Proof. First we show

$$S_{n-2} \circ Pf(X^{sk}) \supset P_4(X^{sk}). \quad (15)$$

We use induction on the size of the matrix.

The first step is $2n = 6$. We denote by $[i_1, i_2, i_3, i_4]$ the Pfaffian of the sub matrix with the rows and columns i_1, i_2, i_3 and i_4 . Let $f = [i_1, i_2, i_3, i_4] \in P_4(X^{sk})$. We have $\binom{6}{4} = 15$ choices for f . For any of these choices we get the Pfaffian of a 2×2 sub matrix of the form

$$\begin{pmatrix} 0 & x \\ -x & 0 \end{pmatrix},$$

as the coefficient of f in the Pfaffian of the matrix X^{sk} . So if we differentiate the 6×6 Pfaffian with respect to that variable x , we get the 4×4 Pfaffian $f = [i_1, i_2, i_3, i_4]$.

Now we assume that equation (15) holds for the generic skew symmetric $(2n-2) \times (2n-2)$ matrix, and we want to show it holds for the $2n \times 2n$ generic skew symmetric matrix. The Pfaffian of the skew symmetric $2n \times 2n$ matrix X^{sk} can be computed recursively as

$$Pf(X^{sk}) = \sum_{i=2}^{i=2n} (-1)^i x_{1i}^{sk} Pf(X_{1i}^{sk}), \quad (16)$$

where X_{1i}^{sk} denotes the matrix X^{sk} with both the first and the i -th rows and columns removed. So X_{1i}^{sk} is a $(2n-2) \times (2n-2)$ matrix and equation (15) holds for it. So for each choice of $[i_1, i_2, i_3, i_4]$ of the matrix X_{1i}^{sk} we can find $n-3$ variables of X_{1i}^{sk} such that differentiating $Pf(X_{1i}^{sk})$ with respect to those variables gives us $[i_1, i_2, i_3, i_4]$. If we call those variable a_1, \dots, a_{n-3} . Using equation (16) if we add x_{1i}^{sk} to our set of $n-3$ variables we will have a set of $n-2$ variables such that if we differentiate $Pf(X^{sk})$ with respect to those $n-2$ variables we will get $[i_1, i_2, i_3, i_4]$. Since we could write the recursive formula for the Pfaffian with respect to any other row or column, the result follows.

For the opposite inclusion to (15) we have

$$W \subset (\text{Ann}(Pf(X^{sk})))_2 \subset (\text{Ann}(P_4(X^{sk})))_2.$$

But we have shown in Lemma 4.6 that

$$W = (\text{Ann}(P_4(X^{sk})))_2.$$

So we have

$$(\text{Ann}(Pf(X^{sk})))_2 = (\text{Ann}(P_4(X^{sk})))_2.$$

By Remark 1.4 we have

$$(\text{Ann}(Pf(X^{sk})))_2 = \text{Ann}(S_{n-2} \circ (Pf(X^{sk}))).$$

Hence we have

$$S_{n-2} \circ Pf(X^{sk}) = P_4(X^{sk}).$$

□

Recall that we denote by $P_{2k}(X^{sk})$ the vector subspace of R^{sk} spanned by the $2k$ -Pfaffian minors of X^{sk} [Definition 4.2]. Analogous to Lemma 1.3 we have

Lemma 4.8. *For $1 \leq k \leq n-1$ we have*

$$S_k \circ (Pf(X^{sk})) = P_{2n-2k}(X^{sk}). \quad (17)$$

Proof. First we want to show

$$S_k \circ (Pf(X^{sk})) \subset P_{2n-2k}(X^{sk}).$$

We use induction on k . First, let $k = 1$, we want to show

$$S_1 \circ (Pf(X^{sk})) \subset P_{2n-2}(X^{sk}).$$

We need to show for any monomial $y_{ij} \in S_1$ we have

$$y_{ij} \circ (Pf(X^{sk})) \subset P_{2n-2}(X^{sk}).$$

It is enough to show the above inclusion holds for y_{12} . Using equation (16) we have

$$y_{12} \circ (Pf(X^{sk})) = y_{12} \circ \sum_{i=2}^{i=2n} (-1)^i x_{1i}^{sk} Pf(X_{\hat{1}\hat{i}}^{sk}) = Pf(X_{\hat{1}\hat{2}}^{sk}) + \sum_{i=3}^{i=2n} (-1)^i x_{1i}^{sk} Pf(X_{\hat{1}\hat{i}}^{sk}) \in P_{2n-2}(X^{sk}).$$

So we have

$$S_1 \circ (Pf(X^{sk})) \subset P_{2n-2}(X^{sk}).$$

Now assume $S_k \circ (Pf(X^{sk})) \subset P_{2n-2k}(X^{sk})$. We want to show

$$S_{k+1} \circ (Pf(X^{sk})) \subset P_{2n-2k-2}(X^{sk}).$$

We have

$$S_{k+1} \circ (Pf(X^{sk})) = S_1 \circ (S_k \circ (Pf(X^{sk}))) \subset S_1 \circ (P_{2n-2k}(X^{sk})) \subset P_{2n-2k-2}(X^{sk}).$$

For the other inclusion, we again use induction on k . First we show the inclusion holds for $k = 1$. Let $\eta \in P_{2n-2}(X^{sk})$ be a $(2n-2) \times (2n-2)$ Pfaffian minor of X^{sk} . Corresponding to η there exists a 2×2 matrix of the form

$$\begin{pmatrix} 0 & x \\ -x & 0 \end{pmatrix},$$

where x is not in the $2n-2$ rows and columns of η . Now if we differentiate the Pfaffian of X^{sk} with respect to x we will get η . So we have $\eta \in S_1 \circ (Pf(X^{sk}))$.

Now assume $P_{2n-2k}(X^{sk}) \subset S_k \circ (Pf(X^{sk}))$, we have

$$P_{2n-2k-2}(X^{sk}) \subset S_1 \circ (P_{2n-2k}(X^{sk})) \subset S_1 \circ (S_k \circ (Pf(X^{sk}))) = S_{k+1} \circ (Pf(X^{sk})).$$

Thus by induction the equality holds. □

Recall that (W) is the ideal of S^{sk} spanned by degree 2 elements of type (a), (b) and (c) as in Definition 4.5. The following proposition is analogous to Proposition 2.10.

Proposition 4.9. *For the $2n \times 2n$ generic skew symmetric matrix X^{sk} we have*

$$(W)_n = \text{Ann}(Pf(X^{sk})) \cap S_n^{sk} \quad (18)$$

Proof. Let $2 \leq k \leq n$. By Remark 1.4 and Lemma 4.8 we have

$$(1) \quad W \circ Pf(X^{sk}) = 0 \iff W \circ S_{n-2}^{sk} Pf(X^{sk}) = 0 \iff W \circ P_4(X^{sk}) = 0.$$

$$(2) \quad (\text{Ann}(Pf(X^{sk}))) \cap S_2 = W \Rightarrow S_{k-2}W \circ (S_{n-k} \circ Pf(X^{sk})) = 0.$$

$$\Rightarrow S_{k-2}(W) \circ P_{2k}(X^{sk}) = 0.$$

$$\Rightarrow (W)_k \circ P_{2k}(X^{sk}) = 0. \quad (\text{By Remark 1.4})$$

Therefore for all k , $2 \leq k \leq n$ we have

$$(W)_k \subset \text{Ann}(P_{2k}(X^{sk})) \cap S_k^{sk}. \quad (19)$$

We need to show

$$(W)_n \supset \text{Ann}(Pf(X^{sk})) \cap S_n^{sk}. \quad (20)$$

We use induction on n . For $n = 1, 2$, we have the 2×2 and 4×4 skew symmetric matrices and the equality is easy to see. Now we want to show that the proposition holds for $n = 3$.

We use the Remark 2.7. Let η be a binomial in $\text{Ann}(Pf(X^{sk})) \cap S_3^{sk}$. Without loss of generality we can write

$$\eta = y_{12}y_{34}y_{56} - y_{\sigma(1)\sigma(2)}y_{\sigma(3)\sigma(4)}y_{\sigma(5)\sigma(6)}.$$

Where $\sigma \in S_6$, $\text{sgn}(\sigma) = 1$ and we have $\sigma(1) < \sigma(3) < \sigma(5)$ and $\sigma(1) < \sigma(2)$, $\sigma(3) < \sigma(4)$ and $\sigma(5) < \sigma(6)$.

If the two terms of the binomial η have a common factor then without loss of generality we can assume that the common factor is y_{12} so we can write η as

$$\eta = y_{12}(y_{34}y_{56} - y_{\sigma(3)\sigma(4)}y_{\sigma(5)\sigma(6)})$$

□

But by the definition of $(W)_3$ the monomial $y_{34}y_{56} - y_{\sigma(3)\sigma(4)}y_{\sigma(5)\sigma(6)}$ is included in W since it is of the form (c). So we have $\eta \in (W)_3$.

Now assume that the two terms of η , i.e. $y_{12}y_{34}y_{56}$ and $y_{\sigma(1)\sigma(2)}y_{\sigma(3)\sigma(4)}y_{\sigma(5)\sigma(6)}$ do not have any common factor. We can add and subtract another term of the Pfaffian $\tau = y_{\beta(1)\beta(2)}y_{\beta(3)\beta(4)}y_{\beta(5)\beta(6)}$ such that β is a permutation in S_6 and we have $\beta(1) < \beta(3) < \beta(5)$ and $\beta(1) < \beta(2)$, $\beta(3) < \beta(4)$ and $\beta(5) < \beta(6)$. and τ has one common factor with $y_{12}y_{34}y_{56}$ and one common factor with $y_{\sigma(1)\sigma(2)}y_{\sigma(3)\sigma(4)}y_{\sigma(5)\sigma(6)}$. Without loss of generality we can take $\beta(5) = 5$, $\beta(6) = 6$ and $\beta(1) = \sigma(1)$, $\beta(2) = \sigma(2)$. So we have

$$\eta - \tau + \tau = \eta - y_{\sigma(1)\sigma(2)}y_{\beta(3)\beta(4)}y_{5,6} + y_{\sigma(1)\sigma(2)}y_{\beta(3)\beta(4)}y_{5,6}.$$

Hence we have

$$\eta = y_{5,6}(y_{12}y_{34} - y_{\sigma(1)\sigma(2)}y_{\beta(3)\beta(4)}) + y_{\sigma(1)\sigma(2)}(y_{\beta(3)\beta(4)}y_{5,6} - y_{\sigma(3)\sigma(4)}y_{\sigma(5)\sigma(6)}).$$

But by the definition of W we know that $y_{12}y_{34} - y_{\sigma(1)\sigma(2)}y_{\beta(3)\beta(4)}$ and $y_{\beta(3)\beta(4)}y_{5,6} - y_{\sigma(3)\sigma(4)}y_{\sigma(5)\sigma(6)}$ are both elements of W of type (c). So we have $\eta \in (W)_3$.

When n is larger than 3 by the induction assumption we can assume that the proposition holds for all integers $k \leq n - 1$. Again we use the Remark 2.7. Assume $b = b_1 + b_2$ is

of degree n . If the two terms, b_1 and b_2 are monomials in S^{sk} and have a common factor l , i.e. $b_1 = la_1$ and $b_2 = la_2$, then $b = l(a_1 + a_2)$ where a_1 and a_2 are of degree at most $n - 1$. Now by the induction assumption the proposition holds for the binomial $a_1 + a_2$, i.e. $a_1 + a_2 \in W_{n-1}$ hence we have

$$b = l(a_1 + a_2) \in l(W)_{n-1} \subset (W)_n.$$

If the two terms, b_1 and b_2 do not have any common factor then with the same method as above we can rewrite the binomial b by adding and subtracting a term m of degree n , which has a common factor m_1 with b_1 and a common factor m_2 with b_2 , now we will have

$$b_1 + b_2 = b_1 + m + b_2 - m = m_1(c_1 + m') + m_2(c_2 - m''),$$

where $b_1 = m_1c_1$, $m = m_1m' = m_2m''$ and $b_2 = m_2c_2$. Now $c_1 + m'$ and $c_2 - m''$ are of degree at most $n - 1$ so by the induction assumption we have

$$b_1 + b_2 = m_1(c_1 + m') + m_2(c_2 - m'') \in (W)_n.$$

This completes the induction step and the proof of the proposition.

Corollary 4.10. *For $1 \leq k \leq n$ we have*

$$(W)_k = \text{Ann}(Pf(X^{sk})) \cap S_k^{sk}$$

We also have $(W)_{n+1} = S_{n+1}^{sk}$.

Proof. Using equation (20) we only need to show that

$$\text{Ann}(Pf(X^{sk})) \cap S_k^{sk} \subset (W)_k$$

By Remark 1.4 and Lemma 4.8 we have

$$(\text{Ann}(Pf(X^{sk})))_k = (\text{Ann}(S_{n-k} \circ Pf(X^{sk})))_k = (\text{Ann}(P_{2k}(X^{sk})))_k$$

Now if we label the $2k \times 2k$ Pfaffians of X^{sk} by f_1, \dots, f_s we have

$$\text{Ann}(P_{2k}(X^{sk}))_k = \text{Ann} \langle f_1, \dots, f_s \rangle_k = \left(\bigcap_{i=1}^{i=s} (\text{Ann}(f_i)) \right)_k$$

Let R^i denote the ring of variables of f_i and $W(i)$ the f_i variables that are involved. By Proposition 4.9 we have

$$(W(i))_k = \text{Ann}(f_i) \cap S_k^i$$

So we have

$$\text{Ann}(Pf(X^{sk})) \cap S_k^{sk} \subset (W)_k$$

To prove the second part, it is easy to see that every monomial of degree larger than n will be unacceptable, of type (a) or (b) in W . So we have $(W)_{n+1} = S_{n+1}^{sk}$. \square

Theorem 4.11. *Let X^{sk} be a generic skew symmetric $2n \times 2n$ matrix. Then the apolar ideal $\text{Ann}(Pf(X^{sk}))$ is the ideal W generated in degree 2.*

Proof. This follows directly from Proposition 4.9 and Corollary 4.10. \square

Corollary 4.12. *Let X^{sk} be a $2n \times 2n$ generic skew symmetric matrix. We have*

$$cr(Pf(X^{sk})) \geq 2^{2n-2}$$

Proof. Using Ranestad-Schreyer Proposition, Corollary 4.4 and Theorem 4.11 we have

$$cr(Pf(X^{sk})) \geq \frac{1}{2} \dim(S^{sk}/\text{Ann}(Pf(X^{sk}))) = \frac{1}{2}(2^{2n-1}) = 2^{2n-2}. \quad (21)$$

\square

Remark 4.13. For $n \geq 5$ it can be easily seen that the lower bound for the cactus rank given by Corollary 4.12 is larger than $l_{diff} = \binom{2n}{2t_0}$, where $t_0 = \lfloor n/2 \rfloor$.

Theorem 4.14. *Let X^s be a generic symmetric $2n \times 2n$ matrix. Then the apolar ideal $\text{Ann}(Hf(X^s))$ is generated in degree 2. And the inequality (19) also holds for $(Hf(X^s))$.*

Proof. By the definition of the Hafnian, it is easy to see that none of the diagonal elements appear in $Hf(X^s)$. so for $1 \leq i \leq 2n$ we have

$$y_{ii} \circ Hf(X^s) = 0$$

Hence without loss of generality we can restrict our discussion to the case where X^s is a generic zero-diagonal symmetric matrix. Now by changing the Pfaffians to Hafnians,

the proof follows directly from the proofs that we have for the Pfaffian of generic skew symmetric matrix.

□

5 Gröbner bases

In Section 2 we have shown that for A a generic $n \times n$ matrix $\text{Ann}(\det(A)) = (\mathcal{P}_D + \mathcal{U}_D)$. In [LS], R. Laubenbacher and I. Swanson give a Gröbner bases for the ideal of 2×2 permanents of a matrix. In this section we first review their result (Theorem 5.2) and then state our result for the ideal $\text{Ann}(\det(A))$ and prove it independently (Theorem 5.3).

Definition 5.1. ([LS], page 197) Let $D = (d_{ij})$ be the matrix of the differential operators as defined in section 1. A monomial order on the d_{ij} is diagonal if for any square submatrix of D , the leading term of the permanent (or of the determinant) of that submatrix is the product of the entries on the main diagonal. An example of such an order is the lexicographic order defined by:

$$d_{ij} < d_{kl} \text{ if and only if } l > j \text{ or } l = j \text{ and } k > i.$$

Throughout this section we use an arbitrary lexicographic diagonal ordering.

Theorem 5.2. ([LS], page 197) *The following collection G of polynomials is a minimal reduced Gröbner basis for \mathcal{P}_D , with respect to any diagonal ordering:*

- (1) *The subpermanents $d_{ij}d_{kl} + d_{kj}d_{il}$, $i < k, j < l$;*
- (2) *$d_{i_1 j_1} d_{i_1 j_2} d_{i_2 j_3}$, $i_1 > i_2, j_1 < j_2 < j_3$;*
- (3) *$d_{i_1 j_1} d_{i_2 j_2} d_{i_2 j_3}$, $i_1 > i_2, j_1 < j_2 < j_3$;*
- (4) *$d_{i_1 j_1} d_{i_2 j_1} d_{i_3 j_2}$, $i_1 < i_2 < i_3, j_1 > j_2$;*
- (5) *$d_{i_1 j_1} d_{i_2 j_2} d_{i_3 j_2}$, $i_1 < i_2 < i_3, j_1 > j_2$;*
- (6) *$d_{i_1 j_1}^{e_1} d_{i_2 j_2}^{e_2} d_{i_3 j_3}^{e_3}$, $i_1 < i_2 < i_3, j_2 > j_3, e_1 e_2 e_3 = 2$.*

Monomials of type (2), (3), (4), (5) and (6) in the above theorem are in the ideal generated by all unacceptable monomials.

Theorem 5.3. *The collection of unacceptable degree 2 monomials and 2×2 subpermanents of D , form a Gröbner basis for $\text{Ann}(\det(A))$ with respect to any diagonal ordering.*

Proof. We will denote \mathcal{U}_D and \mathcal{P}_D by \mathcal{U}, \mathcal{P} respectively in the following, where D is understood.

The elements of $(\mathcal{U} + \mathcal{P})$ generate $\text{Ann}(\det(A))$. Since \mathcal{U} is a set of monomials, it is already Gröbner. We use Buchberger's algorithm to find a Gröbner basis for $\mathcal{P} + \mathcal{U}$. We have one of the following cases:

a) Let F and G be distinct permanents of D . Let $F = a_{ik}a_{jl} + a_{il}a_{jk}$ and $G = a_{uz}a_{vw} + a_{uw}a_{vz}$ be two permanents in \mathcal{P} .

$$F = \text{perm} \begin{pmatrix} a_{ik} & a_{il} \\ a_{jk} & a_{jl} \end{pmatrix}.$$

and

$$G = \text{perm} \begin{pmatrix} a_{uz} & a_{uw} \\ a_{vz} & a_{vw} \end{pmatrix}.$$

Let $f_1 = a_{ik}a_{jl}$ be the leading term of F , and $g_1 = a_{uz}a_{vw}$ be the leading term of G with respect to the given diagonal ordering. Denote the least common multiple of f_1 and g_1 by h_{ij} . Let

$$S(F, G) = (h_{ij}/f_1)F - (h_{ij}/g_1)G = a_{uz}a_{vw}a_{il}a_{jk} - a_{ik}a_{jl}a_{uw}a_{vz}.$$

Now using the multivariate division algorithm, reduce all the $S(F, G)$ relative to the set of all permanents. When there is no common factor in the initial terms of F and G the reduction is zero, as one can use F and G again as we show. First we reduce $S(F, G)$ dividing by $F \in \mathcal{P}$, so we will have

$$S(F, G) + a_{uw}a_{vz}(a_{ik}a_{jl} + a_{il}a_{jk}) = a_{uz}a_{vw}a_{il}a_{jk} + a_{uw}a_{vz}a_{il}a_{jk}.$$

Then we reduce the result using G this time, so we will have

$$a_{uz}a_{vw}a_{il}a_{jk} + a_{uw}a_{vz}a_{il}a_{jk} - a_{il}a_{jk}(a_{uz}a_{vw} + a_{uw}a_{vz}) = 0.$$

So we have shown for all pairs F, G of distinct permanents of D , the S -polynomials $S(F, G)$ reduces to zero with respect to \mathcal{P} .

b) Let $F = a_{ik}a_{jl} + a_{il}a_{jk}$ and $G = a_{ik}a_{jm} + a_{im}a_{jk}$ be two permanents which their initial terms have a common factor. We have

$$S(F, G) = a_{il}a_{jk}a_{jm} - a_{im}a_{jk}a_{jl} \in \mathcal{U}.$$

c) Let $F = a_{im}a_{jn} + a_{in}a_{jm}$ be a permanent and $M = a_{tk}a_{tl}$ be an unacceptable monomial. We have

$$S(F, M) = a_{tk}a_{tl}a_{jm}a_{in} \in \mathcal{U}.$$

d) Let $F = a_{il}a_{jm} - a_{im}a_{jl}$ be a permanent and $M = (a_{kn})^2$ be an unacceptable monomial. We have

$$S(F, M) = a_{im}a_{jl}(a_{kn})^2 \in \mathcal{U}.$$

e) Let $F = a_{il}a_{jm} - a_{im}a_{jl}$ be a permanent and $M = (a_{il})^2$ be an unacceptable monomial which has a common factor with the initial term of F . We have

$$S(F, M) = a_{il}a_{im}a_{jl} \in \mathcal{U}.$$

f) Let $F = a_{il}a_{jm} - a_{im}a_{jl}$ be a permanent and $M = a_{jn}a_{kn}$ be an unacceptable monomial. We have

$$S(F, M) = a_{im}a_{jl}a_{jn}a_{kn} \in \mathcal{U}.$$

So the generating set $\mathcal{P} + \mathcal{U}$ is a Gröbner basis by Buchberger's algorithm.

□

6 Discussion of connected sum

Definition 6.1. ([MS]) A polynomial F in r variables is a connected sum if we can write $F = F' + F''$ with F' and F'' in r' and r'' variables, where $r' + r'' = r$.

Let A be a generic 2×2 matrix, we can write the determinant A is a sum of two polynomials in complementary number of variables.

Proposition 6.2. (Buczyńska, Buczyński, Teitler ([BBT])) *If a form F of degree d is a connected sum, then the apolar ideal has a minimal generator in degree d . (The converse does not hold.)*

In particular, since the generic determinant and permanent of size $n \geq 3$ have their annihilating ideals generated in degree 2, therefore they are not connected sums. This is also true for the Pfaffian of skew symmetric matrices and Hafnian of symmetric matrices of size $n \geq 6$.

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